Some Harmonic Number Identities involving certain Reciprocals

M.J. Kronenburg

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Abstract

Some finite series of harmonic numbers involving certain reciprocals are evaluated. Products of such reciprocals are expanded in a sum of the individual reciprocals, leading to a computer program. A list of examples is provided.

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1 Definitions and Basic Identities

The generalized harmonic numbers used in this paper are:

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m} \tag{1.1}$$

from which follows that $H_0^{(m)} = 0$. The traditional harmonic numbers are:

$$H_n = H_n^{(1)} \tag{1.2}$$

A well known identity is [2, 3, 7, 8]:

$$\sum_{k=1}^{n} \frac{1}{k} H_k = \frac{1}{2}(H_n^2 + H_n^{(2)}) \tag{1.3}$$

and [3, 4]:

$$\sum_{k=0}^{n} \frac{1}{k+1} H_k = \frac{1}{2}(H_{n+1}^2 - H_{n+1}^{(2)}) \tag{1.4}$$

and [5]:

$$\sum_{k=1}^{n} \frac{1}{k} H_{n-k} = H_n^2 - H_n^{(2)} \tag{1.5}$$

$$\sum_{k=0}^{n} \frac{1}{k+1} H_{n-k} = H_{n+1}^2 - H_{n+1}^{(2)} \tag{1.6}$$

When $a \leq b$ are two integers and $\{x_k\}$ and $\{y_k\}$ are two sequences of complex numbers, and $\{s_k\}$ the sequence of complex numbers defined by:

$$s_k = \sum_{i=a}^{k} x_i \tag{1.7}$$

then there is the following summation by parts formula [4]:

$$\sum_{k=a}^{b-1} x_k y_k = s_{b-1} y_b - \sum_{k=a}^{b-1} s_k (y_{k+1} - y_k) \tag{1.8}$$

2 Harmonic Number Identities with a Reciprocal

Theorem 2.1. *For nonnegative integer n and integer $p > 0$:*

$$\begin{aligned}\sum_{k=0}^{n} \frac{1}{k+p} H_k = H_{n+p}(H_{n+1} + H_{p-1}) - \frac{1}{2}[(H_{n+1} + H_{p-1})^2 + H_{n+1}^{(2)} + H_{p-1}^{(2)}]\\ - \sum_{k=0}^{p-2} \frac{1}{n+k+2} H_k\end{aligned} \tag{2.1}$$

Proof. Summation by parts (1.8) with $x_k = 1/(k+p)$ and $y_k = H_k$ yields:

$$\sum_{k=1}^{n} \frac{1}{k+p} H_k = (H_{n+p} - H_p) H_{n+1} - \sum_{k=1}^{n} \frac{1}{k+1} (H_{k+p} - H_p) \tag{2.2}$$

Using:

$$H_{k+p} = H_k + \sum_{s=1}^{p} \frac{1}{k+s} \tag{2.3}$$

and for $s > 1$:

$$\frac{1}{(k+s)(k+1)} = \frac{1}{s-1} (\frac{1}{k+1} - \frac{1}{k+s}) \tag{2.4}$$

yields:

$$\frac{1}{k+1} H_{k+p} = \frac{1}{k+1} H_k + \frac{1}{(k+1)^2} + \sum_{s=2}^{p} \frac{1}{s-1} (\frac{1}{k+1} - \frac{1}{k+s}) \tag{2.5}$$

Performing the summation over n and using (1.4) yields:

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k+p} H_k = H_{n+p} H_{n+1} - \frac{1}{2}(H_{n+1}^2 + H_{n+1}^{(2)}) - H_p + 1\\ + \sum_{s=1}^{p-1} \frac{1}{s} (H_{n+s+1} - H_{n+1} - H_{s+1} + 1)\end{aligned} \tag{2.6}$$

Using

$$H_{n+s+1} - H_{n+1} = \sum_{k=1}^{s} \frac{1}{n+k+1} \tag{2.7}$$

and changing the order of summation over s and k:

$$\begin{aligned}
\sum_{s=1}^{p-1} \frac{1}{s} \sum_{k=1}^{s} \frac{1}{n+k+1} &= \sum_{k=1}^{p-1} \frac{1}{n+k+1} \sum_{s=k}^{p-1} \frac{1}{s} \\
&= \sum_{k=1}^{p-1} \frac{1}{n+k+1} (H_{p-1} - H_{k-1}) \qquad (2.8) \\
&= H_{p-1}(H_{n+p} - H_{n+1}) - \sum_{k=0}^{p-2} \frac{1}{n+k+2} H_k
\end{aligned}$$

and using $H_{s+1} = H_s + 1/(s+1)$ and (1.3) and $1/(s(s+1)) = 1/s - 1/(s+1)$ yields the theorem. □

Theorem 2.2. *For nonnegative integer n and integer $p > 0$:*

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{k+p} H_{n-k} = H_{n+1}(H_{n+p} - H_{p-1}) - \frac{1}{2}[H_{n+1}^2 - H_{n+p}^2 + H_{n+1}^{(2)} + H_{n+p}^{(2)}] \\
- \sum_{k=0}^{p-2} \frac{1}{n+k+2} H_k
\end{aligned} \tag{2.9}$$

Proof. Summation by parts (1.8) with $x_k = 1/(n+p-k)$ and $y_k = H_k$ yields:

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{k+p} H_{n-k} &= \sum_{k=1}^{n} \frac{1}{n+p-k} H_k \\
&= (H_{n+p-1} - H_{p-1}) H_{n+1} - \sum_{k=1}^{n} \frac{1}{k+1} (H_{n+p-1} - H_{n+p-k-1})
\end{aligned} \tag{2.10}$$

Using:

$$H_{n+p-k-1} = H_{n-k} + \sum_{s=1}^{p-1} \frac{1}{n+s-k} \tag{2.11}$$

and for $s > 0$:

$$\frac{1}{(n+s-k)(k+1)} = \frac{1}{n+s+1} \left(\frac{1}{k+1} + \frac{1}{n+s-k}\right) \tag{2.12}$$

yields:

$$\frac{1}{k+1} H_{n+p-k-1} = \frac{1}{k+1} H_{n-k} + \sum_{s=1}^{p-1} \frac{1}{n+s+1} \left(\frac{1}{k+1} + \frac{1}{n+s-k}\right) \tag{2.13}$$

Performing the summation over n and using (1.6) yields:

$$\sum_{k=1}^{n} \frac{1}{k+p} H_{n-k} = H_{n+p-1} - H_{p-1} H_{n+1} + H_{n+1}^2 - H_{n+1}^{(2)} - H_n \\ + \sum_{s=1}^{p-1} \frac{1}{n+s+1} (H_{n+s-1} + H_{n+1} - H_{s-1} - 1) \tag{2.14}$$

Using

$$\sum_{s=1}^{p-1} \frac{1}{n+s+1} (H_{n+1} - 1) = (H_{n+1} - 1)(H_{n+p} - H_{n+1}) \tag{2.15}$$

and $H_{n+s-1} = H_{n+s} - 1/(n+s)$ and $1/((n+s)(n+s+1)) = 1/(n+s) - 1/(n+s+1)$ and with (1.6):

$$\begin{aligned} \sum_{s=0}^{p-2} \frac{1}{n+s+2} H_{n+s+1} &= \sum_{s=0}^{n+p-1} \frac{1}{k+1} H_k - \sum_{s=0}^{n} \frac{1}{k+1} H_k \\ &= \frac{1}{2} (H_{n+p}^2 - H_{n+p}^{(2)} - H_{n+1}^2 + H_{n+1}^{(2)}) \end{aligned} \tag{2.16}$$

yields the theorem. □

Theorem 2.3. *For nonnegative integer n and integer $0 \le p \le n$:*

$$\sum_{k=p+1}^{n} \frac{1}{k-p} H_k = \frac{1}{2} [(H_{n-p+1} + H_p)^2 + H_{n-p+1}^{(2)} + H_p^{(2)}] - H_{n+1} (H_p + \frac{1}{n-p+1}) \\ + \sum_{k=0}^{p-1} \frac{1}{n-p+k+2} H_k \tag{2.17}$$

Proof. Summation by parts (1.8) with $x_k = 1/(k-p)$ and $y_k = H_k$ yields:

$$\begin{aligned} \sum_{k=p+1}^{n} \frac{1}{k-p} H_k &= H_{n+1} H_{n-p} - \sum_{k=p+1}^{n} \frac{1}{k+1} H_{k-p} \\ &= H_{n+1} H_{n-p} - \sum_{k=1}^{n-p} \frac{1}{k+p+1} H_k \end{aligned} \tag{2.18}$$

The last sum is (2.1) with p replaced by $p+1$ and n by $n-p$, which yields the theorem. □

Theorem 2.4. *For nonnegative integer n and integer $0 \le p \le n$:*

$$\sum_{k=p+1}^{n} \frac{1}{k-p} H_{n-k} = H_{n-p}^2 - H_{n-p}^{(2)} \tag{2.19}$$

Proof.

$$\sum_{k=p+1}^{n} \frac{1}{k-p} H_{n-k} = \sum_{k=0}^{n-p-1} \frac{1}{k+1} H_{n-p-k-1} \tag{2.20}$$

The last sum is (1.6) with n replaced by $n-p-1$, which yields the theorem. □

3 Products of Reciprocals

A finite product of these reciprocals with different p's can be written as a sum of the individual reciprocals. The formula for two reciprocals is, where $p_1 \neq p_2$:

$$\begin{aligned} \frac{1}{(k+p_1)(k+p_2)} &= \frac{1}{p_1-p_2} \frac{(k+p_1)-(k+p_2)}{(k+p_1)(k+p_2)} \\ &= \frac{1}{p_1-p_2}\left(\frac{1}{k+p_2} - \frac{1}{k+p_1}\right) \end{aligned} \tag{3.1}$$

The formula for three reciprocals is, where $p_1 \neq p_2 \neq p_3$:

$$\begin{aligned} \frac{1}{(k+p_1)(k+p_2)(k+p_3)} &= \frac{1}{p_1-p_2}\Big[\Big(\frac{1}{p_2-p_3} - \frac{1}{p_1-p_3}\Big)\frac{1}{k+p_3} \\ &\quad - \frac{1}{p_2-p_3}\frac{1}{k+p_2} + \frac{1}{p_1-p_3}\frac{1}{k+p_1}\Big] \end{aligned} \tag{3.2}$$

The recursion formula for m reciprocals in terms of the formula for $m-1$ reciprocals is:

$$\prod_{i=1}^{m-1} \frac{1}{k+p_i} = \sum_{i=1}^{m-1} \alpha_i \frac{1}{k+p_i} \tag{3.3}$$

$$\begin{aligned} \prod_{i=1}^{m} \frac{1}{k+p_i} &= \sum_{i=1}^{m-1} \alpha_i \frac{1}{k+p_m} \frac{1}{k+p_i} \\ &= -\sum_{i=1}^{m-1} \alpha_i \frac{1}{p_i-p_m} \frac{1}{k+p_i} \\ &\quad + \frac{1}{k+p_m} \sum_{i=1}^{m-1} \alpha_i \frac{1}{p_i-p_m} \end{aligned} \tag{3.4}$$

This recursion formula means that starting with $m=1$ and $\alpha_1 = 1$, in each pass for certain $m > 1$ the α_i for $i = 1 \cdots m-1$ are divided by $p_m - p_i$, after which α_m is minus the sum of the new α_i for $i = 1 \cdots m-1$. This way the recursion formula reduces to a double iteration, and it is also clear from this that for $m > 1$:

$$\sum_{i=1}^{m} \alpha_i = 0 \tag{3.5}$$

When the α_i have been computed, each individual reciprocal can be summed using the appropriate formula in the previous section, where the following substitutions are made:

$$H_{n+p}^{(m)} = H_{n+1}^{(m)} + \sum_{k=2}^{p} \frac{1}{(n+k)^m} \tag{3.6}$$

$$H_{n-p+1}^{(m)} = H_{n+1}^{(m)} - \sum_{k=0}^{p-1} \frac{1}{(n-k+1)^m} \tag{3.7}$$

After these substitutions the coefficient of H_{n+1}^2 in the formula for each individual reciprocal is identical, and therefore, by equation (3.5), the coefficient of H_{n+1}^2 in the resulting formula for $m > 1$ is zero, which means that for these products of these reciprocals only terms linear in harmonic numbers remain.

4 Examples

$$\sum_{k=0}^{n} \frac{1}{k+1} H_k = \frac{1}{2}(H_{n+1}^2 - H_{n+1}^{(2)}) \tag{4.1}$$

$$\sum_{k=0}^{n} \frac{1}{k+2} H_k = \frac{1}{2}(H_{n+1}^2 - H_{n+1}^{(2)}) + \frac{1}{n+2} H_{n+1} - \frac{n+1}{n+2} \tag{4.2}$$

$$\sum_{k=0}^{n} \frac{1}{k+3} H_k = \frac{1}{2}(H_{n+1}^2 - H_{n+1}^{(2)}) + \frac{2n+5}{(n+2)(n+3)} H_{n+1} - \frac{(n+1)(7n+20)}{4(n+2)(n+3)} \tag{4.3}$$

$$\sum_{k=1}^{n} \frac{1}{k} H_k = \frac{1}{2}(H_n^2 + H_n^{(2)}) \tag{4.4}$$

$$\sum_{k=2}^{n} \frac{1}{k-1} H_k = \frac{1}{2}(H_{n+1}^2 + H_{n+1}^{(2)}) - \frac{2n+1}{n(n+1)} H_{n+1} + \frac{n}{n+1} \tag{4.5}$$

$$\sum_{k=3}^{n} \frac{1}{k-2} H_k = \frac{1}{2}(H_{n+1}^2 + H_{n+1}^{(2)}) - \frac{3n^2-1}{(n-1)n(n+1)} H_{n+1} + \frac{7n^2-n-2}{4n(n+1)} \tag{4.6}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} H_k = H_{n+1}^{(2)} - \frac{1}{n+1} H_{n+1} \tag{4.7}$$

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} H_k = \frac{n+1}{n+2} - \frac{1}{n+2} H_{n+1} \tag{4.8}$$

$$\sum_{k=2}^{n} \frac{1}{k(k-1)} H_k = \frac{2n+1}{n+1} - \frac{1}{n} H_{n+1} \tag{4.9}$$

$$\sum_{k=3}^{n} \frac{1}{(k-1)(k-2)} H_k = \frac{9n^2+5n-2}{4n(n+1)} - \frac{1}{n-1} H_{n+1} \tag{4.10}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} H_k = \frac{1}{2} H_{n+1}^{(2)} - \frac{1}{2(n+1)(n+2)} H_{n+1} - \frac{n+1}{2(n+2)} \tag{4.11}$$

$$\sum_{k=2}^{n} \frac{1}{(k+1)k(k-1)} H_k = \frac{5n+3}{4(n+1)} - \frac{1}{2n(n+1)} H_{n+1} - \frac{1}{2} H_{n+1}^{(2)} \tag{4.12}$$

$$\sum_{k=3}^{n} \frac{1}{k(k-1)(k-2)} H_k = \frac{2n^2+2n-1}{4n(n+1)} - \frac{1}{2n(n-1)} H_{n+1} \tag{4.13}$$

$$\begin{aligned}\sum_{k=2}^{n} \frac{1}{(k+2)(k+1)k(k-1)} H_k &= \frac{23n^2+57n+28}{36(n+1)(n+2)} \\ &\quad - \frac{1}{3n(n+1)(n+2)} H_{n+1} - \frac{1}{3} H_{n+1}^{(2)}\end{aligned} \tag{4.14}$$

$$\begin{aligned}\sum_{k=3}^{n} \frac{1}{(k+1)k(k-1)(k-2)} H_k &= \frac{1}{6} H_{n+1}^{(2)} - \frac{1}{3(n-1)n(n+1)} H_{n+1} \\ &\quad - \frac{2n^2+1}{12n(n+1)}\end{aligned} \tag{4.15}$$

$$\sum_{k=0}^{n} \frac{1}{k+1} H_{n-k} = H_{n+1}^2 - H_{n+1}^{(2)} \tag{4.16}$$

$$\sum_{k=0}^{n} \frac{1}{k+2} H_{n-k} = H_{n+1}^2 - H_{n+1}^{(2)} - \frac{n}{n+2} H_{n+1} \tag{4.17}$$

$$\sum_{k=0}^{n} \frac{1}{k+3} H_{n-k} = H_{n+1}^2 - H_{n+1}^{(2)} - \frac{3n^2+7n-2}{2(n+2)(n+3)} H_{n+1} - \frac{n+1}{(n+2)(n+3)} \tag{4.18}$$

$$\sum_{k=1}^{n} \frac{1}{k} H_{n-k} = H_n^2 - H_n^{(2)} \tag{4.19}$$

$$\sum_{k=2}^{n} \frac{1}{k-1} H_{n-k} = H_{n+1}^2 - H_{n+1}^{(2)} - \frac{2(2n+1)}{n(n+1)} H_{n+1} + \frac{2(3n^2+3n+1)}{n^2(n+1)^2} \tag{4.20}$$

$$\sum_{k=3}^{n} \frac{1}{k-2} H_{n-k} = H_{n+1}^2 - H_{n+1}^{(2)} - \frac{2(3n^2-1)}{(n-1)n(n+1)} H_{n+1} + \frac{2(6n^4-3n^2+1)}{(n-1)^2 n^2 (n+1)^2} \tag{4.21}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} H_{n-k} = \frac{n-1}{n+1} H_{n+1} - \frac{n-1}{(n+1)^2} \tag{4.22}$$

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} H_{n-k} = \frac{n}{n+2} H_{n+1} \tag{4.23}$$

$$\sum_{k=2}^{n}\frac{1}{k(k-1)}H_{n-k}=\frac{n-2}{n}H_{n+1}-\frac{(n-2)(2n+1)}{n^2(n+1)} \tag{4.24}$$

$$\sum_{k=3}^{n}\frac{1}{(k-1)(k-2)}H_{n-k}=\frac{n-3}{n-1}H_{n+1}-\frac{(n-3)(3n^2-1)}{(n-1)^2n(n+1)} \tag{4.25}$$

$$\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}H_{n-k}=\frac{n^2+3n-2}{4(n+1)(n+2)}H_{n+1}-\frac{3n-1}{4(n+1)^2} \tag{4.26}$$

$$\sum_{k=2}^{n}\frac{1}{(k+1)k(k-1)}H_{n-k}=\frac{n^2+n-4}{4n(n+1)}H_{n+1}-\frac{4n^3+3n^2-9n-4}{4n^2(n+1)^2} \tag{4.27}$$

$$\sum_{k=3}^{n}\frac{1}{k(k-1)(k-2)}H_{n-k}=\frac{n^2-n-4}{4n(n-1)}H_{n+1}-\frac{(n^2-2n-1)(5n^2+3n-4)}{4(n-1)^2n^2(n+1)} \tag{4.28}$$

$$\begin{aligned}\sum_{k=2}^{n}\frac{1}{(k+2)(k+1)k(k-1)}H_{n-k}&=\frac{n^3+3n^2+2n-12}{18n(n+1)(n+2)}H_{n+1}\\&\quad-\frac{7n^3+18n^2-25n-12}{36n^2(n+1)^2}\end{aligned} \tag{4.29}$$

$$\begin{aligned}\sum_{k=3}^{n}\frac{1}{(k+1)k(k-1)(k-2)}H_{n-k}&=\frac{n^3-n-12}{18(n-1)n(n+1)}H_{n+1}\\&\quad-\frac{9n^5+9n^4-41n^3-81n^2+32n+24}{36(n-1)^2n^2(n+1)^2}\end{aligned} \tag{4.30}$$

5 Computer Program

The Mathematica® [9] program used to compute the expressions is given below.

```
HarmNumPlus[p_,m_]:=HarmonicNumber[n+1,m]+Sum[1/(n+k)^m,{k,2,p}]
HarmNumMinus[p_,m_]:=HarmonicNumber[n+1,m]-Sum[1/(n-k+1)^m,{k,0,p-1}]
HarmSumPPos[p_,d_]:=Simplify[
 HarmNumPlus[p,1](HarmonicNumber[n+1]+HarmonicNumber[p-1])
 -1/2((HarmonicNumber[n+1]+HarmonicNumber[p-1])^2
 +HarmonicNumber[n+1,2]+HarmonicNumber[p-1,2])
 -Sum[1/(n+k+2)HarmonicNumber[k],{k,0,p-2}]
 -Sum[1/(k+p)HarmonicNumber[k],{k,0,d-1}]]
HarmSumPNeg[p_,d_]:=Simplify[
 1/2((HarmNumMinus[p,1]+HarmonicNumber[p])^2
 +HarmNumMinus[p,2]+HarmonicNumber[p,2])
 -HarmonicNumber[n+1](HarmonicNumber[p]+1/(n-p+1))
 +Sum[1/(n-p+k+2)HarmonicNumber[k],{k,0,p-1}]
 -Sum[1/(k-p)HarmonicNumber[k],{k,p+1,d-1}]]
HarmSumP[p_,d_]:=If[p<=0,HarmSumPNeg[-p,d],HarmSumPPos[p,d]]
```

```
HarmSumQPos[p_,d_]:=Simplify[
 HarmonicNumber[n+1](HarmNumPlus[p,1]-HarmonicNumber[p-1])
 -1/2(HarmonicNumber[n+1]^2-HarmNumPlus[p,1]^2
 +HarmonicNumber[n+1,2]+HarmNumPlus[p,2])
 -Sum[1/(n+k+2)HarmonicNumber[k],{k,0,p-2}]
 -Sum[1/(k+p)HarmNumMinus[k+1,1],{k,0,d-1}]]
HarmSumQNeg[p_,d_]:=Simplify[
 HarmNumMinus[p+1,1]^2-HarmNumMinus[p+1,2]
 -Sum[1/(k-p)HarmNumMinus[k+1,1],{k,p+1,d-1}]]
HarmSumQ[p_,d_]:=If[p<=0,HarmSumQNeg[-p,d],HarmSumQPos[p,d]]
HarmTable[m_]:=Table[HarmonicNumber[n+1,i],{i,m}]
HarmSumPQ[s_Integer,f_]:=Module[{d,u,t=HarmTable[2]},
 d=If[s<=0,-s+1,0];u=Factor[CoefficientArrays[f[s,d],t]];
 u[[1]]+Dot[u[[2]],t]+Dot[Dot[u[[3]],t],t]]
HarmSumPQ[s_,f_]:=Module[{facs,d,u,funs=0,l=Length[s],t=HarmTable[2]},
 facs=Table[0,{l}];facs[[1]]=1;Do[Do[facs[[j]]/=(s[[i]]-s[[j]]);
 facs[[i]]-=facs[[j]],{j,1,i-1}],{i,2,l}];
 d=Min[s];d=If[d<=0,-d+1,0];Do[funs+=facs[[i]]f[s[[i]],d],{i,1,l}];
 u=Factor[CoefficientArrays[funs,t]];u[[1]]+Dot[u[[2]],t]]
HarmonicSumP[s_]:=If[Length[s]==1,HarmSumPQ[s[[1]],HarmSumP],
 HarmSumPQ[s,HarmSumP]]
HarmonicSumQ[s_]:=If[Length[s]==1,HarmSumPQ[s[[1]],HarmSumQ],
 HarmSumPQ[s,HarmSumQ]]

(* Compute some examples *)
HarmonicSumP[3]//TraditionalForm
HarmonicSumP[{2,1,0,-1}]//TraditionalForm
HarmonicSumQ[{-2}]//TraditionalForm
HarmonicSumQ[{0,-1,-2}]//TraditionalForm
```

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